Stable bundle extensions on elliptic Calabi–Yau threefolds

Björn Andreas [a,*], Gottfried Curio [b]

[a] *Institut für Mathematik und Informatik, Freie Universität Berlin, Arnimallee 14, 14195 Berlin, Germany*
[b] *Arnold-Sommerfeld-Center for Theoretical Physics, Department für Physik, Ludwig-Maximilians-Universität München, Theresienstr. 37, 80333 München, Germany*

Abstract

We construct stable bundle extensions on elliptically fibered Calabi–Yau threefolds. We show that these bundles can solve the topological anomaly constraint in heterotic string theory without the need for invoking background five-branes.
© 2007 Elsevier B.V. All rights reserved.

Keywords: Calabi–Yau threefolds; Topological anomaly; Heterotic string theory

1. Introduction

In the present paper we consider the class of elliptically fibered Calabi–Yau threefolds $\pi: X \to B$ with a section σ and construct stable vector bundles V of vanishing first Chern class using the method of bundle extensions. For a choice of data the bundles satisfy

$$c_2(TX) = c_2(V)$$

and so qualify as physical gauge bundles for heterotic string theory compactification.

In contrast, the widely used spectral cover construction [1–3] gives stable vector bundles on elliptic fibrations. These bundles solve the generalized anomaly constraint $c_2(TX) - c_2(V) = [W]$ with $[W]$ an effective curve class (cf. below, Section 7). This mismatch causes two problems in physical model building: firstly, for $[W] \neq 0$ it prevents the model from being interpreted as a nonlinear sigma model; secondly, for $[W] \neq 0$, and even for $[W] = 0$ while $V \neq TX$, it leads to singular three-forms in the heterotic anomaly equation. As a consequence, it is more difficult to solve the anomaly equation because a nontrivial string theory H-field has to be taken into account. A detailed discussion of these issues is given in the physical companion paper [4].

The paper is organized as follows. In Section 2, a general outline of the construction method is given and the necessary steps for proving the stability of a given non-split extension are described. We consider extensions of $W_q \otimes \mathcal{O}_X(pD)$ by $U_p \otimes \mathcal{O}_X(-qD)$ where U_p and W_q are given stable vector bundles of vanishing first Chern class. D is a divisor in X, chosen such that the resulting vector bundle has a trivial determinant. The main result of this

---

section is Lemma 2.3, which gives a sufficient condition for the bundle not to be destabilized by certain subsheaves. In Section 3, we restrict to extensions V of $\mathcal{O}_X(nD)$ by $\pi^*E \otimes \mathcal{O}_X(-D)$ where E is a given stable vector bundle on the base of the Calabi–Yau threefold. The main result of this section is Lemma 3.1; it provides conditions for choosing such an extension V non-split. In Section 4, we prove the stability of a pull-back bundle π^*E, assuming that the base of the Calabi–Yau space is given by the Enriques surface (Lemma 4.1). We then prove that V is stable in a specific region of the Kähler cone (Proposition 4.2). In Section 5, we generalize the results of the previous section to elliptic Calabi–Yau threefolds with ample K_B^{-1}. In Section 6, we consider extensions of stable spectral cover bundles V_n and prove the stability for extensions of $\mathcal{O}_X(n\pi^*\alpha)$ by $V_n \otimes \mathcal{O}_X(-\pi^*\alpha)$ where $\alpha \in H^2(B, \mathbb{Z})$. In Section 7, we give explicit solutions to the topological anomaly constraint imposed by heterotic string theory.

Throughout the paper we use the notation $c_i = c_i(B)$.

2. Method of construction

Let $\pi: X \to B$ be an elliptically fibered Calabi–Yau threefold with a section σ. Except in Section 6, we will consider B to be either an Enriques surface or a surface with ample K_B^{-1} such as the Hirzebruch surface F_r with $r = 0, 1$ or the del Pezzo surface dP_k with $k = 0, \ldots, 8$.

We consider vector bundles V of rank $p + q$ on X defined as nontrivial extensions of bundles U_p and W_q.

Assumption 2.1. We assume that U_p and W_q are stable vector bundles both of vanishing first Chern class.

Furthermore, U_p and W_q should be suitably twisted by line bundles such that V has vanishing first Chern class

$$0 \to U_p \otimes \mathcal{O}_X(-qD) \to V \to W_q \otimes \mathcal{O}_X(pD) \to 0$$

where $D = x\sigma + \pi^*\alpha$. To discuss stability we will choose as polarization $J = z\sigma + \pi^*H$ where H (chosen in the integral cohomology) is in the Kähler cone $\mathcal{C}_B$ of the base B and $z \in \mathbb{R}^{>0}$. For an elliptically fibered Calabi–Yau space X one finds, via the Nakai-Moishezon criterion, that J in the Kähler cone $\mathcal{C}_X$ of X under the following conditions [5]

$$J \in \mathcal{C}_X \Longleftrightarrow z > 0, \qquad H - zc_1 \in \mathcal{C}_B.$$

The following lemma is obvious.

Lemma 2.2. *Two necessary conditions for V to be stable are*

(i) $DJ^2 > 0$ *or equivalently* $\mu(U_p \otimes \mathcal{O}_X(-qD)) < 0$
(ii) $W_q \otimes \mathcal{O}_X(pD)$ *is not a subbundle of V, i.e. the exact sequence defining V can be chosen non-split.*

For the rest of this section let us assume that the non-split condition can be satisfied.

To discuss the required steps for proving the stability of V consider the following diagram of exact sequences

$$
\begin{array}{ccccccccc}
& & 0 & & 0 & & 0 & & \\
& & \uparrow & & \uparrow & & \uparrow & & \\
0 & \to & P = \bar{P} \otimes \mathcal{O}_X(-qD) & \to & V/V'_{r+s} & \to & T = \bar{T} \otimes \mathcal{O}_X(pD) & \to & 0 \\
& & \uparrow & & \uparrow & & \uparrow & & \\
0 & \to & U_p \otimes \mathcal{O}_X(-qD) & \overset{i}{\to} & V & \overset{j}{\to} & W_q \otimes \mathcal{O}_X(pD) & \to & 0 \\
& & \uparrow & & \uparrow & & \uparrow & & \\
0 & \to & F_r \otimes \mathcal{O}_X(-qD) & \to & V'_{r+s} & \to & G_s \otimes \mathcal{O}_X(pD) & \to & 0 \\
& & \uparrow & & \uparrow & & \uparrow & & \\
& & 0 & & 0 & & 0 & &
\end{array}
$$

with $F_r \otimes \mathcal{O}_X(-qD) = i^{-1}V'_{r+s}$ and $G_s \otimes \mathcal{O}_X(pD) = j(V'_{r+s})$ of ranks $0 \le r \le p$ and $0 \le s \le q$ for a subsheaf V'_{r+s} of V.

In the following we will discuss the required steps for proving stability of V. In total we have to consider all subsheaves V'_{r+s} of V with $0 \le r \le p$ and $0 \le s \le q$. However, we can exclude certain cases.

Step 1: We first note that the cases $(0, 0)$, $(p, 0)$ and (p, q) do not occur as destabilizing subsheaves. The cases $(0, 0)$ and (p, q) are ruled out as we only have to consider subsheaves V'_{r+s} of rank $r+s$ with $0 < r+s < p+q$. Note

that $r = 0$ implies $F_r = 0$ as U_p does not have a non-zero subsheaf of vanishing rank (the same holds correspondingly for $s = 0$). Moreover, we can assume [10, Lemma 4.5] that the quotient V/V'_{r+s} is torsion free and so cases with $r = p$ need not be considered since $(U_p \otimes \mathcal{O}_X(-qD))/(F_p \otimes \mathcal{O}_X(-qD))$ is a torsion sheaf.

Step 2: To prove the stability of V we need to show that $\mu(V'_{r+s}) < 0$ for all $0 < r + s < p + q$ with $0 \leq r \leq p - 1$ and $0 \leq s \leq q$. We first note that

$$(r + s)\mu(V'_{r+s}) = (ps - qr)DJ^2 + r\mu(F_r) + s\mu(G_s)$$

and the discussion depends on the sign of $(ps - qr)$.

$(ps - qr) < 0$: As $DJ^2 > 0$ by assumption and $\mu(F_r) < 0$, respectively, $\mu(G_s) < 0$ we get in this case $\mu(V'_{r+s}) < 0$.

$(ps - qr) > 0$: We have $\mu(F_r) < 0$ and $\mu(G_s) < 0$ for $0 < r < p$ and $0 < s < q$. Further if $s = q$ then $\mu(G_q) \leq 0$ according to [10, Lemma 4.3] So we get the following subcases:

(i) $\mu(F_r) < 0, \mu(G_s) < 0$
(ii) $\mu(F_r) < 0, \mu(G_q) < 0$
(iii) $\mu(F_r) < 0, \mu(G_q) = 0$
(iv) $r = 0, \mu(G_q) < 0$
(v) $r = 0, \mu(G_q) = 0$.

(i)–(iv): We have to solve $\mu(V'_{r+s}) < 0$ for z, together with $DJ^2 > 0$ for all subsheaves V'_{r+s} for $0 < r+s < p+q$ (cf. Propositions 4.2, 5.2 and 6.1).

(v): This case has to be excluded as $\mu(V'_{0+q}) = (p)DJ^2 > 0$.

The following result gives a condition when subsheaves $V'_{0+q} \cong G_q \otimes \mathcal{O}_X(pD)$ do not destabilize V because they do not exist. So the cases (iv) and (v) would then be excluded if one could show the corresponding assertion about the f-map, however, following this line of argumentation we will actually exclude case (v) below (cf. Lemma 4.4).

Lemma 2.3. *Let $U := U_p \otimes \mathcal{O}_X(-qD)$, $W := W_q \otimes \mathcal{O}_X(pD)$ and $G := G_q \otimes \mathcal{O}_X(pD)$. A sufficient condition for V not to be destabilized by a subsheaf G of W is given by the injectivity of the map*

$$\mathrm{Ext}^1(W, U) \xrightarrow{f} \mathrm{Ext}^1(G, U).$$

Proof. For the convenience of the reader we provide a proof of this standard fact from homological algebra. We first ask when is it possible that a map $G \to W$ lifts to a map $G \to V$. To see this consider

$$\to \mathrm{Hom}(G, V) \to \mathrm{Hom}(G, W) \to \mathrm{Ext}^1(G, U)$$

showing that the obstruction to lifting an element of $\mathrm{Hom}(G, W)$ to an element of $\mathrm{Hom}(G, V)$ lies in $\mathrm{Ext}^1(G, U)$. We have a commutative diagram

$$\begin{array}{ccc} \mathrm{Hom}(W, W) & \xrightarrow{\partial} & \mathrm{Ext}^1(W, U) \\ \downarrow & & \downarrow \\ \mathrm{Hom}(G, W) & \to & \mathrm{Ext}^1(G, U) \end{array}$$

with $\partial(1) = \xi$ the extensions class. So we conclude that a non-zero element of $\mathrm{Hom}(G, W)$ can be lifted to an element of $\mathrm{Hom}(G, V)$ exactly when the extension class ξ is in the kernel of

$$f: \mathrm{Ext}^1(W, U) \to \mathrm{Ext}^1(G, U)$$

thus if f is injective $f(\xi) \neq 0$ and such a lifting does not exist. $\square$

$(ps - qr) = 0$: This case in principle has to be treated separately in a manner similar to the case $ps - qr > 0$.

Below (Sections 4–6) we will show for extensions of the type $(p, q) = (n, 1)$ the following: the case $ps - qr = 0$ does not occur, the f-map arguments for the case $(0, q)$ can be carried through and the non-split condition can be fulfilled.

Finally, a direct computation gives the Chern classes of V

$$c_1(V) = 0$$
$$c_2(V) = -\frac{1}{2}pq(p+q)D^2 + c_2(U_p) + c_2(W_q)$$
$$c_3(V) = \frac{1}{3}pq(p^2 - q^2)D^3 + 2\left(qc_2(U_p) - pc_2(W_q)\right)D + c_3(U_p) + c_3(W_q).$$

3. Non-split conditions

We will now restrict the general set-up of Section 2 to extensions with U_p given by stable pull-back bundles π^*E (with E a rank n vector bundle on B, stable with respect to an ample class H in B and $c_1(E) = 0$) and $W_q = \mathcal{O}_X$ such that the resulting vector bundle V has rank $m = n + 1$. To prove the stability of π^*E and V will then be our main focus in the subsequent two sections.

The following result provides sufficient conditions for choosing non-split extensions if U_p is given by π^*E.

Let $y := mx$ and $E_1 := R^1\pi_*\mathcal{O}_X(-y\sigma) \otimes E \otimes \mathcal{O}_B(-m\alpha)$ and $E_2 := \pi_*\mathcal{O}_X(-y\sigma) \otimes \mathcal{O}_B(-m\alpha) \otimes E$ (the expressions E_1 and E_2 occur below in the Leray spectral sequence).

Lemma 3.1. *Let E be a H-stable rank n vector bundle on a rational surface B with $c_1(E) = 0$, and let $D = x\sigma + \pi^*\alpha$ then an extension of $\mathcal{O}_X(nD)$ by $\pi^*E \otimes \mathcal{O}_X(-D)$ can be chosen non-split for*

  (i) $x > 0$: *if* $(2H - zc_1)\alpha \leq 0$ *and* $\chi(B, E_1) > 0$.
 (ii) $x < 0$: *if* $\chi(B, E_2) < 0$.
(iii) $x = 0$: *if* $\chi(B, E \otimes \mathcal{O}_B(-m\alpha)) < 0$.

Proof. We apply the Leray spectral sequence to $\pi: X \to B$ and use the projection formula giving

$$0 \to H^1(B, E_2) \to H^1(X, \pi^*E \otimes \mathcal{O}_X(-mD)) \to H^0(B, E_1) \to H^2(B, E_2).$$

If $x > 0$ then $\pi_*\mathcal{O}_X(-y\sigma) = 0$ thus

$$H^1(X, \pi^*E \otimes \mathcal{O}_X(-mD)) \cong H^0(B, E_1)$$

now Serre duality on B and $[R^1\pi_*\mathcal{O}_X(-y\sigma)]^* = \pi_*\mathcal{O}_X(y\sigma) \otimes K_B^{-1}$ give

$$H^2(B, E_1)^* = H^0(B, \pi_*\mathcal{O}_X(y\sigma) \otimes \mathcal{O}_B(m\alpha) \otimes E^*)$$

where $\pi_*\mathcal{O}_X(y\sigma) = \mathcal{O}_B \oplus K_B^2 \oplus \cdots \oplus K_B^y$ for $y > 1$ [2, Lemma 4.1] thus

$$H^2(B, E_1)^* = H^0(B, \mathcal{O}_B(m\alpha) \otimes E^*) \oplus H^0\left(B, \bigoplus_{i=2}^y K_B^i \otimes \mathcal{O}_B(m\alpha) \otimes E^*\right).$$

Now the first term vanishes if

$$n\mu(\mathcal{O}_B(m\alpha) \otimes E^*) = (2H - zc_1)\alpha \leq 0$$

all other terms vanish if

$$n\mu(K_B^i \otimes \mathcal{O}_B(m\alpha) \otimes E^*) = -i(2H - zc_1)c_1 + m(2H - zc_1)\alpha \leq 0$$

as $2H - zc_1 \in \mathcal{C}_B$ it follows $(2H - zc_1)c_1 \geq 0$ and so we only have to impose $(2H - zc_1)\alpha \leq 0$. Note that for B, the Enriques surface, this condition becomes $\alpha H \leq 0$. For a surface with K_B^{-1} ample and $H = hc_1$, cf. below, the condition becomes $(2h - z)\alpha c_1 \leq 0$; as $2h - z > 0$ this is equivalent to $\alpha c_1 \leq 0$, i.e., again $\alpha H \leq 0$.

Having $H^2(B, E_1)^* = 0$, we can now apply the Hirzebruch–Riemann–Roch formula and conclude that if $\chi(B, E_1) > 0$ then $H^1(X, \pi^*E \otimes \mathcal{O}_X(-mD))$ is non-zero completing the proof of (i).

(ii) and (iii): If $x < 0$ then $R^1\pi_*\mathcal{O}_X(-y\sigma) = 0$ and the Leray spectral sequence gives

$$H^1(X, \pi^*E \otimes \mathcal{O}_X(-mD)) \cong H^1(B, E_2)$$

thus if $\chi(B, E_2) < 0$ we have $H^1(B, E_2) \neq 0$.

If $x = 0$ then the Leray spectral sequence simplifies (with $\pi_* \mathcal{O}_X = \mathcal{O}_B$)

$$0 \to H^1(B, E \otimes \mathcal{O}_B(-m\alpha)) \to H^1(X, \pi^* E \otimes \mathcal{O}_X(-mD)) \to$$

and a sufficient condition for the first space to be non-zero is $\chi(B, E \otimes \mathcal{O}_B(-m\alpha)) < 0$. $\quad\square$

Let us state the explicit expressions for $\chi(B, E_1)$, $\chi(B, E_2)$ and $\chi(B, E \otimes \mathcal{O}_B(-m\alpha))$. For $y = mx > 0$ we note that $R^1\pi_* \mathcal{O}_X(-y\sigma) = K_B^1 \oplus K_B^{-1} \oplus \cdots \oplus K_B^{1-y}$ for $y > 1$ [2, Lemma 5.16] and

$$ch(R^1\pi_* \mathcal{O}_X(-y\sigma)) = y + A_1 c_1 + A_2 \frac{c_1^2}{2}$$

where we have set $A_1 = -1 + \frac{y(y-1)}{2}$ and $A_2 = 1 + \frac{y(y-1)(2y-1)}{6}$. The Hirzebruch–Riemann–Roch formula gives

$$\chi(B, E_1) = y\left(n - c_2(E) + \frac{nm^2}{2}\alpha^2\right) + A_3 \frac{n}{2} c_1^2 - A_4 nm\alpha c_1$$

where $A_3 = \frac{y(y^2-1)}{3}$ and $A_4 = -1 + \frac{y^2}{2}$. For $y = mx < 0$ we obtain

$$ch(\pi_* \mathcal{O}_X(-y\sigma)) = -y - A_1 c_1 - A_2 \frac{c_1^2}{2}$$

and the Hirzebuch–Riemann–Roch formula yields

$$\chi(B, E_2) = -y\left(n - c_2(E) + \frac{nm^2}{2}\alpha^2\right) - A_3 \frac{n}{2} c_1^2 + A_4 nm\alpha c_1.$$

If $x = 0$ we find

$$\chi(B, E \otimes \mathcal{O}_B(-m\alpha)) = n - c_2(E) + \frac{nm}{2}\alpha(m\alpha - c_1).$$

Remark 3.2. Note that for $x > 0$ the case $\alpha = c_1$ does not lead to $H^1(X, \pi^* E \otimes \mathcal{O}_X(-mD)) \neq 0$. E being supposed to be stable, one has $H^0(B, E) = 0$; the same holds for the slope-zero stable bundle $K_B \otimes E$. However, if E is a H-semistable vector bundle of zero slope on the Enriques surface and $H^0(B, E) \neq 0$ then for $x > 0$ and $\alpha = c_1$ one can choose an extension of $\mathcal{O}_X(nD)$ by $\pi^* E \otimes \mathcal{O}_X(-D)$ to be non-split.

Lemma 3.3. *For $x > 0$, the condition $\mu(K_B^{1-y} \otimes \mathcal{O}_B(-m\alpha) \otimes E) > 0$ is necessary for the existence of a non-split extension of $\mathcal{O}_X(nD)$ by $\pi^* E \otimes \mathcal{O}_X(-D)$.*

Proof. From Lemma 3.1 we have for $x > 0$ that

$$H^1(X, \pi^* E \otimes \mathcal{O}_X(-mD)) \cong H^0(B, R^1\pi_* \mathcal{O}_X(-y\sigma) \otimes E \otimes \mathcal{O}_B(-m\alpha)).$$

If $\mu(K_B^{1-y} \otimes \mathcal{O}_B(-m\alpha) \otimes E) \leq 0$ then we have $H^0(B, E_1) = 0$ and so the extension splits. $\quad\square$

Corollary 3.4. *For $x \neq 0$ and B the Enriques surface the following relation necessarily holds when V is stable*

$$x \cdot (\alpha H) < 0.$$

Proof. We apply Lemma 2.2. For $x < 0$ from the condition

$$DJ^2 = x\sigma(H - zc_1)^2 + z(2H - zc_1)\alpha\sigma > 0$$

it follows that $H\alpha > 0$ (for the cases with K_B^{-1} ample and $H = hc_1$ we get $\alpha c_1 > 0$). For $x > 0$ the non-split condition gives

$$n\mu(K_B^{1-y} \otimes \mathcal{O}_B(-m\alpha) \otimes E) = (y-1)(2H - zc_1)c_1 - m(2H - zc_1)\alpha > 0$$

becoming $-2m\alpha H > 0$ for the Enriques surface. $\quad\square$

4. Stable extensions on the Enriques CY space

In this section we consider a Calabi–Yau space X, elliptically fibered over an Enriques surface B, cf. Appendix. Given a stable vector bundle E of rank $n \geq 2$ with $c_1(E) = 0$ on B, we will construct rank $n + 1$ vector bundles of trivial determinant on X as non-split extensions of $\mathcal{O}_X(nD)$ by $\pi^* E_n \otimes \mathcal{O}_X(-D)$ with $D = x\sigma + \pi^* \alpha$ and prove that V is stable in a region of the Kähler cone of X. For this we show first that $\pi^* E$ is stable on X provided that E is stable on the Enriques surface. For the existence of stable vector bundles on Enriques surfaces see [8,9]. The fact that $\pi^* E$ is stable on Calabi–Yau threefolds elliptically fibered over the Enriques surface has been used previously in [6].

Lemma 4.1. $\pi^* E$ *is (semi)stable with respect to* $J = z\sigma + \pi^* H$ *on* X *if* E *is (semi)stable on* B *with respect to* H *and with* $c_1(E) = 0$.

Proof. Let $\mathcal{F}$ be a subsheaf of $\pi^* E$ where we can assume that $\pi^* E / \mathcal{F}$ is torsion free [10, Lemma 4.5]; so we have $0 \to \mathcal{F}|_\sigma \to E$ and $c_1(\mathcal{F}|_\sigma) H < 0$ (for semistability ≤ 0). Similarly we get $0 \to \mathcal{F}|_F \to \mathcal{O}_F^r$ thus $\deg(\mathcal{F}|_F) \leq 0$ as $\mathcal{O}_F^r$ is semistable (where $r := rk\, E$). We conclude that $c_1(\mathcal{F}) =: -A\sigma + \pi^* \lambda$ has $A \geq 0$ and $\lambda H < 0$ and $c_1(\mathcal{F}) J^2 = -AH^2\sigma + 2z\lambda H\sigma < 0$ (with $<$ replaced by $\leq$ for semistability). $\quad\square$

In the following proposition we construct stable bundles V of vanishing first Chern class (recall that $xa < 0$ for $x \neq 0$ by Corollary 3.4, where $a = \alpha H$).

Proposition 4.2. *Let* V *be a rank* $n + 1$ *vector bundle on* $\pi: X \to B$ *defined by a non-split extension*

$$0 \to \pi^* E \otimes \mathcal{O}_X(-D) \to V \to \mathcal{O}_X(nD) \to 0$$

with E *a rank* n, H-*stable bundle with* $c_1(E) = 0$ *on an Enriques surface* B *and* $D = x\sigma + \pi^* \alpha$ *and* $a := \alpha H$. *Then* V *is stable with respect to* $J = z\sigma + \pi^* H$ *for* $|x| < |a|$ *and*

(i) $x > 0$ *and* $\frac{nx}{1-na} \frac{H^2}{2} < z < \frac{nx}{-na} \frac{H^2}{2}$

(ii) $x < 0$ *and* $\frac{-nx}{na} \frac{H^2}{2} < z < \frac{-nx}{na-1} \frac{H^2}{2}$.

Proof. We have to consider now the following diagram of exact sequences

$$
\begin{array}{ccccccccc}
& & 0 & & 0 & & 0 & & \\
& & \uparrow & & \uparrow & & \uparrow & & \\
0 & \to & P = \bar{P} \otimes \mathcal{O}_X(-D) & \to & V/V'_{r+1} & \to & T = \bar{T} \otimes \mathcal{O}_X(nD) & \to & 0 \\
& & \uparrow & & \uparrow & & \uparrow & & \\
0 & \to & \pi^* E \otimes \mathcal{O}_X(-D) & \xrightarrow{i} & V & \xrightarrow{j} & \mathcal{O}_X \otimes \mathcal{O}_X(nD) & \to & 0 \\
& & \uparrow & & \uparrow & & \uparrow & & \\
0 & \to & F_r \otimes \mathcal{O}_X(-D) & \to & V'_{r+1} & \to & G_1 \otimes \mathcal{O}_X(nD) & \to & 0 \\
& & \uparrow & & \uparrow & & \uparrow & & \\
& & 0 & & 0 & & 0 & &
\end{array}
$$

In view of the discussion in Section 2 we have to prove the stability of V for $0 < r + 1 < n + 1$ with $0 \leq r \leq n - 1$ and show $\mu(V'_{r+1}) < 0$. We have

$$(r + 1)\mu(V'_{r+1}) = (n - r)DJ^2 + r\mu(F_r) + \mu(G_1)$$

with $\mu(F_r) < 0$ by stability of $\pi^* E$ and $\mu(G_1) \leq 0$ by [10, Lemma 4.3]. We first note that cases with $(n - r) = 0$ do not occur as $r \neq n$. We are left with discussing the cases $(n - r) > 0$ leading to the following subcases:

(i) $\mu(F_r) < 0, \mu(G_1) < 0$
(ii) $\mu(F_r) < 0, \mu(G_1) = 0$
(iii) $r = 0, \mu(G_1) < 0$
(iv) $r = 0, \mu(G_1) = 0$.

(ii) and (iii): We must solve the following inequalities for z simultaneously (thereby solving (i))

$$(n - r)DJ^2 + r\mu(F_r) < 0, \qquad (n - r)DJ^2 + \mu(G_1) < 0, \qquad DJ^2 > 0$$

where the last inequality assures that $\mu(\pi^*E \otimes \mathcal{O}_X(-D)) < 0$.

By the stability of π^*E we have $c_1(F_r) = -A\sigma + \pi^*\lambda$ with $-A \leq 0$ and $\lambda H < 0$ and $c_1(G_1) = -D_2$ with $D_2 = B'\sigma + \pi^*\bar{\beta}$ an effective divisor. We set $-\bar{\beta} = \beta$ such that $c_1(G_1) = -B'\sigma + \pi^*\beta$. The slopes of F_r and G_1 are given by

$$r\mu(F_r) = -AH^2\sigma + 2z\lambda H\sigma, \qquad \mu(G_1) = -B'H^2\sigma + 2z\beta H\sigma.$$

We estimate the first two inequalities which give

$$nDJ^2 + r\mu(F_r) < 0, \qquad nDJ^2 + \mu(G_1) < 0.$$

To evaluate these expressions it is clearly enough to pose the following conditions to the slopes of F_r and G_1

$$\lambda H = -1 \quad \text{and} \quad A = 0,$$
$$\beta H = -1 \quad \text{and} \quad B' = 0, \quad \text{resp. } \beta H = 0 \quad \text{and} \quad -B' = -1$$

where we have chosen for $-B'$, respectively, βH the worst case, i.e. the biggest values, which are -1. In summary, we have to solve for z the following system (with $a := \alpha H$ and all intersection products taken in B)

$$n(xH^2 + 2za) - 2z < 0$$
$$n(xH^2 + 2za) - H^2 < 0$$
$$xH^2 + 2za > 0.$$

We find for $0 < x < -a$ the condition

$$\frac{nx}{1 - na}\frac{H^2}{2} < z < \frac{-x}{a}\frac{H^2}{2}.$$

If $x = 0$ and $a > 0$ we get $na < 1$ and $z < \frac{1}{na}\frac{H^2}{2}$; as $na \geq n$ we find that in case (ii) we cannot solve the conditions which would exclude a destabilizing subsheaf of V.

Finally, for $-a < x < 0$ we obtain the condition

$$\frac{-x}{a}\frac{H^2}{2} < z < \frac{nx}{1 - na}\frac{H^2}{2}.$$

(iv): This case will be treated in Lemma 4.4 below. We show that potential subsheaves of V of the type V'_{0+1} with $\mu(V'_{0+1}) = nDJ^2 > 0$ do not exist. □

To treat the $(0, 1)$ cases with $\mu(G_1) = 0$ let us first determine the general structure of G_1.

Lemma 4.3. G_1 has the structure $G_1 = \mathcal{O}_X(-D_2) \otimes I_Y$ with $D_2 \geq 0$ and codim $Y \geq 2$.

Proof. For the convenience of the reader we recall the proof of this standard fact. In the $(0, 1)$ case $G_1 \otimes \mathcal{O}_X(nD) = V' \hookrightarrow V_2$ and the torsion sheaf $\bar{T}$ is a quotient of $\mathcal{O}$, so $\bar{T} = \mathcal{O}_Z$ for a subscheme Z of X, and, considering the vertical sequence on the right, $G_1 = I_Z$ where I_Z is the ideal sheaf of Z. Further $D_2J^2 \geq 0$ for $D_2 = c_1(I_Z)$ which comprises the codim $= 1$ components of Z; so $I_Z = \mathcal{O}_X(-D_2) \otimes I_Y$ where $Y \ j : Y \to X$ is the closed immersion of the union of all components of Z of codim ≥ 2. □

This result and the following result apply for any extension of the type

$$0 \to V_n \otimes \mathcal{O}_X(-D) \to V \to \mathcal{O}_X(nD) \to 0$$

on a general Calabi–Yau threefold X.

Lemma 4.4. $V' = I_Y \otimes \mathcal{O}_X(nD)$ with codim $Y \geq 2$ does not occur as a subsheaf of V.

Proof. Let $\bar{V}_n = V_n \otimes \mathcal{O}_X(-D)$. By Lemma 2.3 we have to prove that $f : \text{Ext}^1(\mathcal{O}_X(nD), \bar{V}_n) \to \text{Ext}^1(I_Y(nD), \bar{V}_n)$ is injective. For this consider

$$0 \to I_Y(nD) \to \mathcal{O}_X(nD) \to \mathcal{O}_Y(nD) \to 0$$

taking $\text{Hom}(\cdot, \bar{V}_n)$ yields

$$0 \to \text{Hom}(\mathcal{O}_X(nD), \bar{V}_n) \to \text{Hom}(I_Y(nD), \bar{V}_n) \to \text{Ext}^1(\mathcal{O}_Y(nD), \bar{V}_n) \to$$
$$\to \text{Ext}^1(\mathcal{O}_X(nD), \bar{V}_n) \xrightarrow{f} \text{Ext}^1(I_Y(nD), \bar{V}_n) \to \cdots.$$

We have to show that $\text{Ext}^1(\mathcal{O}_Y(nD), \bar{V}_n) = 0$. Now Serre duality gives $\text{Ext}^1(\mathcal{O}_Y(nD), \bar{V}_n) \cong \text{Ext}^2(\bar{V}_n, \mathcal{O}_Y(nD))^*$ further we have

$$\text{Ext}^2(\bar{V}_n, \mathcal{O}_Y(nD))^* = \text{Ext}^2(\mathcal{O}_X, \bar{V}_n^* \otimes \mathcal{O}_Y(nD))^* = H^2(X, \bar{V}_n^* \otimes \mathcal{O}_Y(nD))^*$$

now as $\mathcal{O}_Y \equiv j_*\mathcal{O}_Y$ we have $H^2(X, \bar{V}_n^* \otimes \mathcal{O}_Y(nD))^* = H^2(X, j_*(j^*(\bar{V}_n^* \otimes \mathcal{O}_X(nD))))^* = H^2(Y, j^*(\bar{V}_n^* \otimes \mathcal{O}_X(nD)))^* = 0$ for codim $Y \geq 2$. $\square$

Remark 4.5. As for $x = 0$ we cannot assure solvability of the numerical slope conditions, one would need to give a condition such that

$$0 \to F_r \otimes O_X(-D) \to V'_{r+1} \to I_Y \otimes \mathcal{O}_X(nD) \to 0$$

for $0 < r < n$ does not occur as potential subsheaf of V.

5. Stable extensions on CY spaces with del Pezzo surface base

In this section we will consider elliptically fibered Calabi–Yau threefolds $\pi : X \to B$ whose base has an ample K_B^{-1}. As in the previous section we consider extensions V of $\mathcal{O}_X(nD)$ by $\pi^*E \otimes \mathcal{O}_X(-D)$. We first note

Lemma 5.1. π^*E is (semi)stable on X with respect to $J = z\sigma + \pi^*H \in \mathcal{C}_X$ (i.e. $H - zc_1 \in \mathcal{C}_B$, so $z < h$) if E is (semi)stable on B with respect to $H = hc_1$ and has $c_1(E) = 0$.

Proof. Let $\mathcal{F}$ be a subsheaf of π^*E where we can assume that $\pi^*E/\mathcal{F}$ is torsion free [10, Lemma 4.5]; so we have $0 \to \mathcal{F}|_\sigma \to E$ and $c_1(\mathcal{F}|_\sigma)H < 0$ (for semistability ≤ 0). Similarly we get $0 \to \mathcal{F}|_F \to \mathcal{O}_F^r$ thus $\deg(\mathcal{F}|_F) \leq 0$ as $\mathcal{O}_F^r$ is semistable (where $r := rk\, E$). Then for $H - zc_1 \in \mathcal{C}_B$ and $c_1(\mathcal{F}) = -A\sigma + \lambda$ with $A \geq 0$ and $\lambda H \leq (Ac_1 + \lambda)H < 0$ (the latter and the following $<$ are $\leq$ for semistability)

$$c_1(\mathcal{F})J^2 = -A(H - zc_1)^2\sigma + z(2H - zc_1)\lambda\sigma < 0. \quad \square$$

We can now proceed as in the previous section and prove the stability of the extension V.

Proposition 5.2. *Let V be a rank $n + 1$ vector bundle on $\pi : X \to B$ defined by a non-split extension*

$$0 \to \pi^*E \otimes \mathcal{O}_X(-D) \to V \to \mathcal{O}_X(nD) \to 0$$

with E a rank n bundle with $c_1(E) = 0$, stable with respect to $H = hc_1$, and $D = x\sigma + \pi^*\alpha$. Then V is stable with respect to $J = z\sigma + \pi^*H$ for $0 < |x| < |a|$ and for z in the following ranges ($\zeta := h - z$)

(i) $0 < x < -a$ and $\frac{nx}{n(xc_1^2-a)+1}H^2 < h^2 - \zeta^2 < \frac{nx}{n(xc_1^2-a)}H^2$
(ii) $-a < x < 0$ and $\frac{nx}{n(xc_1^2-a)}H^2 < h^2 - \zeta^2 < \frac{nx}{n(xc_1^2-a)+1}H^2$.

Proof. The proof is completely parallel to the proof of Proposition 4.2 up to the following consideration. The slopes of F_r, G_1 and the expression for DJ^2 are given by

$$r\mu(F_r) = -A(h - z)^2c_1^2\sigma + z(2h - z)\lambda c_1\sigma$$
$$\mu(G_1) = -B'(h - z)^2c_1^2\sigma + z(2h - z)\beta c_1\sigma$$
$$DJ^2 = x(h - z)^2c_1^2\sigma + z(2h - z)\alpha c_1\sigma$$

inserting these expressions in the estimated inequalities

$$nDJ^2 + r\mu(F_r) < 0, \qquad nDJ^2 + \mu(G_1) < 0$$

and imposing as in Proposition 4.2 the following conditions to the slopes of F_r and G_1

$$\lambda c_1 = -1 \quad \text{and} \quad A = 0,$$
$$\beta c_1 = -1 \quad \text{and} \quad B' = 0, \quad \text{resp. } \beta c_1 = 0 \quad \text{and} \quad -B' = -1$$

we find the following inequalities, which have to be solved for z (here and in the following all intersection products are taken in B)

$$nxh^2c_1^2 + (na - 1 - nxc_1^2)(2h - z)z < 0$$
$$(nx - 1)\,h^2c_1^2 + \left(na + c_1^2 - nxc_1^2\right)(2h - z)\,z < 0$$
$$xh^2c_1^2 + \left(a - xc_1^2\right)(2h - z)\,z > 0.$$

Define $\zeta := h - z$ such that $0 < \zeta < h$. For $0 < x < -a$ we get

$$\frac{nxh^2c_1^2}{n(xc_1^2 - a) + 1} < h^2 - \zeta^2 < \frac{xh^2c_1^2}{xc_1^2 - a}.$$

For $x = 0$ and $a > 0$ we find from the first inequality above

$$(na - 1)(h^2 - \zeta^2) < 0$$

but $(h^2 - \zeta^2) > 0$ and $(na - 1) > 0$.

Finally, for $-a < x < 0$ we get the condition

$$\frac{xh^2c_1^2}{xc_1^2 - a} < h^2 - \zeta^2 < \frac{nxh^2c_1^2}{n(xc_1^2 - a) + 1}. \quad \square$$

6. Stable extensions of spectral cover bundles

In this section we will study non-split extensions of stable spectral cover bundles V_n on $\pi : X \to B$ with B either given by a Hirzebruch surface (or blow-ups of it), a del Pezzo surface or an Enriques surface. We first recall the notion stable spectral cover bundle.

Let X be an elliptically fibered Calabi–Yau threefold with a section σ, let C be an irreducible surface in the linear system $|n\sigma + \eta|$ and $i : C \to X$ the immersion of C into X and let L be a rank one sheaf on C. We say that V_n is a spectral cover bundle of rank n if $V_n = \pi_{1*}(\pi_2^*(i_*L) \otimes \mathcal{P})$ where $\mathcal{P}$ is the Poincaré sheaf on the fiber product $X \times_B X$ and $\pi_{1,2}$ are the respective projections on the first and the second factors. Moreover, V_n is stable with respect to $J = \epsilon J_0 + \pi^*H$ for $0 < \epsilon < \epsilon_0$ as stated in Theorem 7.1 in [2] (we will always assume that ϵ is sufficiently small). Furthermore, note that various aspects of the spectral cover construction have been studied in [11–15]. In particular, in [13] a general statement is provided (without proof) when stable extensions of spectral cover bundles exist.

Let H be an ample divisor in B, we define the minimal H-degree as follows

$$(\Lambda H)_{\min} = \min\{\Lambda \cdot H \mid \Lambda \in H^2(B, \mathbb{Z}) \text{ effective}, \ \Lambda \cdot H > 0\}$$

which will be useful for defining the minimal slope of the subsheaf G_1.

Proposition 6.1. *Let V_n be a stable spectral cover bundle of rank n on $\pi : X \to B$ and let V be defined by a non-split extension*

$$0 \to V_n \otimes \mathcal{O}_X(-D) \to V \to \mathcal{O}_X(nD) \to 0$$

with $D = x\sigma + \pi^*\alpha$ then V is stable with respect to $J = \epsilon\sigma + \pi^*H$ for $x = 0$ and $0 < n\alpha H < (\Lambda H)_{\min}$.

Proof. As in Proposition 4.2 we have to treat the cases (i)–(iv).

(i)–(iii): By stability of V_n we have $c_1(F_r) = -A\sigma + \pi^*\lambda$ with $A > 0$ by Theorem 7.1 of [2] and $c_1(G_1) = -D_2$ with $D_2 = B'\sigma + \pi^*\bar\beta$ an effective divisor. We set $-\bar\beta = \beta$ such that $c_1(G_1) = -B'\sigma + \pi^*\beta$. The slopes of F_r and G_1 and the expression for DJ^2 are given by (we define $\bar H := 2H - \epsilon c_1$)

$$r\mu(F_r) = -AH^2\sigma + \epsilon\,(\lambda + Ac_1)\,\bar H\sigma$$
$$\mu(G_1) = -B'H^2\sigma + \epsilon\,(\beta + Bc_1)\,\bar H\sigma$$
$$DJ^2 = xH^2\sigma + \epsilon\,(\alpha - xc_1)\,\bar H\sigma.$$

As in Proposition 5.2 we have to consider $nDJ^2 + r\mu(F_r) < 0$ which becomes

$$(nx - A)\,H^2\sigma + \epsilon\,(n\alpha + \lambda - (nx - A)c_1)\,\bar H\sigma < 0.$$

For $nDJ^2 + \mu(G_1) < 0$ we have to consider for $-B'$ and $\beta\bar H$ the worst case, i.e., when $|\mu(G_1)|$ is minimal. This will be achieved for either $B' = 0$ and $-\beta\bar H = (-\beta\bar H)_{\min} > 0$ or $-B' = -1$ and $\beta\bar H = 0$.

$$nxH^2\sigma + \epsilon n\,(\alpha - xc_1)\,\bar H\sigma - \epsilon(-\beta\bar H)_{\min}\sigma < 0$$
$$(nx - 1)\,H^2\sigma + \epsilon\,(n\alpha - (nx - 1)c_1)\,\bar H\sigma < 0.$$

For $x > 0$ we cannot solve the first inequality as we have to assume $A = 1$ as the worst case. The cases with $x < 0$ are ruled out by the positivity condition $DJ^2 > 0$.

If $x = 0$ then $DJ^2 = \epsilon(\alpha\bar H) > 0$ implies $\alpha\bar H > 0$ and we get (intersections taken in B)

$$-AH^2 + \epsilon\,(n\alpha + \lambda + Ac_1)\,\bar H < 0$$
$$n\alpha\bar H - (-\beta\bar H) < 0$$
$$-H^2 + \epsilon(n\alpha + c_1)\bar H < 0$$

where the second constraint implies $n\alpha\bar H < (-\beta\bar H)_{\min}$.

Finally, the case (iv) is ruled out by Lemma 4.4. $\quad\square$

Let us give an example for solving $0 < n\alpha\bar H < (-\beta\bar H)_{\min}$.

Example 6.2. Let B be the Enriques surface then $\bar H = 2H$; further let $H, \alpha, \beta \in \Gamma^{1,1}$. We fix a polarization $H = (v, v + 1)$ and set $\beta = -\bar\beta = -(e, f)$. We take $\alpha = (1, -1)$ such that $\alpha H = 1$ and so we have to solve the inequality for n. Then $(-\beta H)_{\min} = v$ and we get solutions for $0 < n < v$.

Finally, we state the conditions such that an extension of $\mathcal{O}_X(n\pi^*\alpha)$ by $V_n \otimes \mathcal{O}_X(-\pi^*\alpha)$ can be chosen non-split.

Lemma 6.3. _Let V_n be a stable spectral cover bundle of rank n and let $\alpha \in H^2(B, \mathbb{Z})$ then an extension of $\mathcal{O}_X(n\pi^*\alpha)$ by $V_n \otimes \mathcal{O}_X(-\pi^*\alpha)$ can be chosen non-split if $\chi(\mathcal{A}, R^1\pi_*V_n|_{\mathcal{A}} \otimes \mathcal{O}_B(-m\alpha)|_{\mathcal{A}}) > 0$ with $\mathcal{A} := \pi(C \cap \sigma)$ and $C \in |n\sigma + \pi^*\eta|$._

Proof. Applying the Leray spectral sequence to $\pi \colon X \to B$ yields

$$0 \to H^1(B, \pi_*V_n \otimes \mathcal{O}_B(-m\alpha)) \to H^1(X, V_n \otimes \mathcal{O}_X(-m\pi^*\alpha))$$
$$\to H^0(B, R^1\pi_*V_n \otimes \mathcal{O}_B(-m\alpha)) \to H^2(B, \pi_*V_n \otimes \mathcal{O}_B(-m\alpha)) \to .$$

For a given spectral cover bundle V_n one has $\pi_*V_n = 0$. At a generic point $b \in B$ one has the stalk $(\pi_*V_n)_b = H^0(F, V_n|_F) = \bigoplus_{i=1}^n H^0(F, \mathcal{O}_F(q_i - p))$ where $p = \sigma F$ is the zero element in the group law on the fiber F over $b \in B$ and q_i are the points at which the spectral cover of V_n intersects F. Now $\mathcal{O}(q_i - p)$ is generically a nontrivial bundle of degree zero which over an elliptic curve admits no global sections. Thus $H^0(F, \mathcal{O}_F(q_i - p)) = 0$ for all i and so $(\pi_*V_n)|_b = 0$. However, since V_n is torsion free, π_*V_n is also torsion free. Thus $(\pi_*V_n)|_b = 0$ for generic $b \in B$ gives $\pi_*V_n = 0$ everywhere. It follows that

$$H^1(X, V_n \otimes \mathcal{O}_X(-m\pi^*\alpha)) \cong H^0(B, R^1\pi_*V_n \otimes \mathcal{O}_B(-m\alpha)).$$

The sheaf $R^1\pi_*V_n$ has support on $\mathcal{A} = \pi(C \cap \sigma)$ of class $\eta - nc_1$ in B and $H^0(B, R^1\pi_*V_n \otimes \mathcal{O}_B(-m\alpha)) \cong H^0(\mathcal{A}, R^1\pi_*V_n|_{\mathcal{A}} \otimes \mathcal{O}_B(-m\alpha)|_{\mathcal{A}})$. The Grothendieck–Riemann–Roch theorem for $\pi \colon X \to B$ gives

$$c_1(R^1\pi_*V_n) = \eta - nc_1.$$

The Riemann–Roch formula gives

$$\chi(\mathcal{A}, R^1\pi_* V_n|_{\mathcal{A}} \otimes \mathcal{O}_B(-m\alpha)|_{\mathcal{A}}) = \frac{3}{2}(\eta - nc_1)^2 - m\alpha(\eta - nc_1)$$

so if $\frac{3}{2}(\eta - nc_1)^2 - m\alpha(\eta - nc_1) > 0$ then $H^0(S, R^1\pi_* V_n|_{\mathcal{A}} \otimes \mathcal{O}_B(-m\alpha)|_{\mathcal{A}}) \neq 0$ and the extension can be chosen to be non-split. Note that irreducibility of the spectral surface (required for stability of V_n) demands $\eta \geq nc_1$. $\square$

7. Physical solutions

A compactification of the perturbative $E_8 \times E_8$ heterotic string on a Calabi–Yau threefold X requires mathematically to construct a pair of stable holomorphic vector bundles (V_1, V_2) of the same slope and trivial determinant on X. Consistency of the physical theory requires the bundles to satisfy the topological constraint

$$c_2(TX) = \sum_i c_2(V_i),$$

the necessary condition for solutions to the heterotic anomaly condition. The spectral cover construction does not lead to vector bundles which solve this topological constraint as $[W] := c_2(TX) - \sum_i c_2(V_i)$ is non-zero. If $[W]$ is non-zero then physically one expects a five-brane to contribute a source term δ_4 (a current that integrates to one in the direction transverse to a single five-brane) to the Bianchi identity for the three-form H. To each five-brane one associates such a four-form delta function source. The class $[W]$ is then the Poincaré dual of an integer sum of all these sources and thus $[W]$ should be integral, representing a class in $H_2(X, \mathbb{Z})$. We will use the same expression $[W]$ for an integral homology class in $H_2(X, \mathbb{Z})$, an integral cohomology class in $H^4(X, \mathbb{Z})$ and the de Rham cohomology class in $H^4_{DR}(X, \mathbb{R})$ (i.e., as $H^p(X, \mathbb{Z}) \to H^p_{DR}(X, \mathbb{R})$ is not injective, integral classes are identified with the images of $H^p(X, \mathbb{Z})$ in $H^p(X, \mathbb{R})$). $[W]$ can be further specified taking into account that supersymmetry requires that five-branes are wrapped on holomorphic curves thus $[W]$ must correspond to the homology class of holomorphic curves. Algebraic classes include negative classes, however, these lead to negative charges, which are unphysical, and so they have to be excluded constraining $[W]$ to be an effective class. Thus for a given Calabi–Yau threefold X the effectivity of $[W]$ constrains the choice of vector bundles V. Consideration of the physical background of a heterotic string compactification [4] reveals that $[W] = 0$ is the favorite case by consistency requirements.

What we will show now is that the vector bundles constructed above allow to solve the topological constraint with $[W] = 0$. To solve the anomaly constraint we set $V_1 = V$ and $V_2 = 0$ we find

$$c_2(X) - c_2(V) = \pi^* w_B \sigma + a_f[F]$$

where $[F]$ denotes the class of a fiber of X. Let V be given by

$$0 \to \bar{V}_n \otimes \mathcal{O}_X(-D) \to V \to \mathcal{O}_X(nD) \to 0$$

where $\bar{V}_n$ will be specified below as either $\pi^* E$ or a spectral bundle V_n, we find (set $c_2(\bar{V}_n) = \pi^* \phi \sigma + \pi^* \omega$ with $\phi \in H^2(B, \mathbb{Z})$ and $\omega \in H^4(B, \mathbb{Z})$)

$$w_B = 12c_1 - \phi + \frac{1}{2}n(n + 1)x(2\alpha - xc_1)$$

$$a_f = c_2 + \frac{1}{2}n(n + 1)\alpha^2 - \omega + 11c_1^2.$$

7.1. The case of B the Enriques surface

Let $\bar{V}_n = \pi^* E$, if B is an Enriques surface the problem of finding solutions to $[W] = 0$ simplifies as the following result shows.

Proposition 7.1. *Let* $\pi : X \to B$ *with* B *an Enriques surface then the physical constraint* $w_B \geq 0$ *implies* $x = 0$.

Proof. For Enriques base we find

$$w_B = \frac{1}{2}n(n+1)x(2\alpha - xc_1) \geq 0$$

$$a_f = 12 + \frac{1}{2}n(n+1)\alpha^2 - c_2(E) \geq 0$$

to solve $w_B \geq 0$ requires $x\alpha \geq 0$ (we already argued from the non-split condition that $\alpha \neq 0, c_1$). So $x\alpha H \geq 0$ contradicting Lemma 3.1. $\square$

If V_2 is nontrivial the argument remains valid as $w_B = \sum_{i=1}^{2} a_i x_i \alpha_i \geq 0$ (with $a_i > 0, a_1 = \frac{n(n+1)}{2}$) gives $w_B H \geq 0$ in contradiction to Lemma 3.1.

But $x = 0$ is the case where the existence of stable bundles could not be assured above.

7.2. The case of B a del Pezzo surface

Let $\pi: X \to B$ an elliptic Calabi–Yau threefold with K_B^{-1} ample. Let $\bar{V}_n = \pi^* E$. In contrast to the case of the Enriques base it is now possible to satisfy $w_B \geq 0$ while having $x \neq 0$. One finds $[W] = 0$ for the choices

$$\alpha = \left(\frac{x^2}{2} - \frac{12}{n(n+1)}\right)\frac{c_1}{x} \Longrightarrow w_B = 0$$

$$c_2(E) = c_2 + 11c_1^2 + \frac{n(n+1)}{2}\alpha^2 \Longrightarrow a_f = 0.$$

For $x > 0$ the non-split condition (cf. Lemma 3.1, (i)) is satisfied if (set $m = n + 1$)

$$x^2 \leq \frac{24}{nm}$$

$$2n + \left(\frac{(3m^3 + m^2)nx^2}{12} + \frac{144}{mx} - \frac{37}{3}n - 20\right)c_1^2 > 24.$$

For instance, for building an $SO(10)$ GUT model without five-branes one can use the twist $D = \sigma - \pi^* c_1/2$ and a rank $n = 3$ bundle E on a base F_r of $c_2(E) = 104$. For another case one may construct an E_6 GUT model without five-branes from using the twist $D = 2\sigma$ and a plane bundle of $c_2(E) = 92$.

7.3. Extensions by spectral bundles

Let $\pi: X \to B$ an elliptic Calabi–Yau threefold with B either given by a Hirzebruch surface (or blow-ups of it), a del Pezzo surface or an Enriques surface. Let $\bar{V}_n = V_n$ a spectral rank n vector bundle has second Chern class equal to [1]

$$c_2(V_n) = \pi^* \eta \, \sigma - \frac{1}{24}\pi^* c_1^2(n^3 - n) + \frac{1}{2}\left(\lambda^2 - \frac{1}{4}\right)n\pi^*\eta(\pi^*\eta - n\pi^* c_1).$$

The condition $c_1(V_n) = 0$ imposes constraints on the spectral data [1]. One finds: if n is even then $\lambda = m + \frac{1}{2}$ and $m \in \mathbb{Z}$. If n is odd one has $\lambda = m$ and $\eta \equiv c_1 \pmod 2$. For this set-up we get

$$w_B = (12c_1 - \eta)$$

$$a_f = c_2 + 11c_1^2 + \frac{1}{2}n(n+1)\alpha^2 + \frac{1}{24}(n^3 - n)c_1^2 - \frac{1}{2}\left(\lambda^2 - \frac{1}{4}\right)n\eta(\eta - nc_1).$$

Now $w_B = 0$ is solved for $\eta = 12c_1$. Then for $a_f = 0$, we have to solve

$$c_2 + c_1^2\left(11 + \frac{n^3 - n}{24} - \frac{1}{2}\left(\lambda^2 - \frac{1}{4}\right)(12 - n)n\right) + \frac{1}{2}n(n+1)\alpha^2 = 0.$$

To give an example let us assume that $B = F_0$. We get for instance $a_f = 0$ for $n = 2$ and $m = 1$ and $\alpha = (1, -11)$. The bundle is stable for $H = (3, 34)$ so $\alpha H = 1$ and $0 < n\alpha H < (-\beta H)_{\min}$ becomes $0 < 2 < 3$, is satisfied. The non-split condition $\frac{3}{2}(\eta - nc_1)^2 - m\alpha(\eta - nc_1) > 0$ is satisfied.

Acknowledgements

Appendix

Let $\pi \colon X \to B$ be an elliptically fibered Calabi–Yau threefold with a section σ and base B given by an Enriques surface, i.e., $h^{1,0}(B) = 0$ and $K_B^2 = \mathcal{O}_B$. We first recall some basic properties of these spaces. B has nontrivial Hodge numbers $h^{1,1}(B) = 10$, $h^{0,0}(B) = h^{2,2}(B) = 1$, so $c_1^2 = 0$ and $c_2 = 12$. Further $\phi c_1 = 0$ for all $\phi \in H^2(B, \mathbb{Z})$ and the intersection form is even [7]. A smooth curve C has $e(C) = -C^2$, and a generic ('unnodal') B has no smooth rational curves. One gets for the middle cohomology $H^2(B, \mathbb{Z}) = \mathbb{Z}^{10} \oplus \mathbb{Z}_2$ with intersection lattice

$$\Gamma^{1,1} \oplus E_8^{(-)} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \oplus E_8^{(-)}$$

(orthogonal decompositions). B is always elliptically fibered over $b = \mathbb{P}^1$. However, two of the fibers, f_1 and f_2, are double fibers: $f = 2 f_i$, which prevents B from having a section and $c_1 = f_1 - f_2$ is not effective.

Let us consider the effective cone. On an unnodal B all irreducible curves C have $C^2 \geq 0$. The integral classes in one of the two components of the cone in $H^2(B, \mathbb{R})$ defined by $C^2 \geq 0$ constitute the effective cone (potentially adding the torsion class c_1 does not matter for this if $C \neq 0$; we will not always mention explicitly this exceptional case). For C nef (i.e. $DC \geq 0$ for all curves C on B) $|C|$ is base-point-free, and C is ample if also $C^2 \geq 6$ [7]. A $C = xa + yf := (x, y) \in \Gamma^{1,1}$ is nef precisely if it is effective in the F_0-sense, i.e., for $x, y \geq 0$.

Furthermore, we note that B can be represented as the quotient of a $K3$ surface by a free involution. The $K3$ can be represented as a double cover $w^2 = f_{4,4}(z_1, z_2)$ of $\mathbb{P}^1_{z_1} \times \mathbb{P}^1_{z_2}$, branched along a curve of bi-degree $(4, 4)$, so elliptically fibered $p_i \colon K3 \to \mathbb{P}^1_{z_i}$. The involution is $(z_1, z_2; w) \to (-z_1, -z_2; -w)$. This shows also two elliptic fibrations of B with the double fibers over 0 and ∞

$$
\begin{array}{ccc}
K3 & \longrightarrow & B \\
p_{K3} \downarrow & & \downarrow p \\
\mathbb{P}^1_{z_1} & \xrightarrow{(\cdot)^2} & \mathbb{P}^1_{z_1}
\end{array}
$$

Note also that in an orbifold limit $T^4/\mathbb{Z}_2$ of $K3$ the involution is $(-1, 1)$ on the complex coordinates (t_1, t_2) of T^4, combined with a shift by a half lattice vector in both directions.

The corresponding $\pi_1(B) = \mathbb{Z}_2$ is inherited by the elliptic Calabi–Yau space X which itself is a quotient by a free involution on $K3 \times T^2$ (it acts as described on $K3$ and as $z \to -z$ on the T^2). The holomorphic two-form Ω_2 of $K3$ being odd, the holomorphic three-form $\Omega_2 \wedge dz$ is preserved, the quotient X being a Calabi–Yau space of vanishing Euler number. Finally, the second Chern class of X can be obtained by a standard computation (cf. [1]) and is given in general by (with $c_i := c_i(B)$)

$$c_2(X) = \pi^* c_2 + 11 \pi^* c_1^2 + 12 \sigma \pi^* c_1.$$

References

[1] R. Friedman, J.W. Morgan, E. Witten, Vector bundles and *F* theory, Comm. Math. Phys. 187 (1997) 679. hep-th/9701162.
[2] R. Friedman, J.W. Morgan, E. Witten, Vector bundles over elliptic fibrations, J. Algebraic Geom. 8 (1999) 279–401. alg-geom/9709029.
[3] R. Donagi, Principal bundles on elliptic fibrations, Asian J. Math. 1 (1997) 214–223. alg-geom/9702002.
[4] B. Andreas, G. Curio, Heterotic models without fivebranes, J. Geom. Phys. 57 (2007) 2136–2145.
[5] B. Andreas, G. Curio, Standard models from heterotic string theory, hep-th/0602247.
[6] R.P. Thomas, Examples of bundles on Calabi–Yau 3-folds for string theory compactifications, Adv. Theor. Math. Phys. 4 (2000) 231–247.
[7] F. Cossec, I. Dolgachev, Enriques Surfaces I, Birkhäuser, 1989.
[8] H. Kim, Moduli spaces of stable vector bundles on enriques surfaces, Nagoya Math. J. 150 (1998) 85.
[9] R. Friedman, Rank two vector bundles over regular elliptic surfaces, Invent Math. 96 (1989) 283–332.
[10] R. Friedman, Algebraic Surfaces and Holomorphic Vector Bundles, Springer, Universitext, 1998.
[11] B. Andreas, On vector bundles and chiral matter in $N = 1$ heterotic compactifications, JHEP 9901 (1999) 011. hep-th/9802202.
[12] G. Curio, Chiral matter and transitions in heterotic string models, Phys. Lett. B 435 (1998) 39. hep-th/9803224.

[13] R. Donagi, B. Ovrut, T. Pantev, D. Waldram, Standard-model bundles, Adv. Theor. Math. Phys. 5 (2002) 563. math.AG/0008010.
[14] R. Donagi, B. Ovrut, T. Pantev, D. Waldram, Spectral involutions on rational elliptic surfaces, Adv. Theor. Math. Phys. 5 (2002) 499. math.AG/0008011.
[15] B. Andreas, D. Hernández Ruipérez, $U(n)$ Vector Bundles on Calabi–Yau Threefolds for String Theory Compactifications, Adv. Theor. Math. Phys. 9 (2006) 253. hep-th/0410170.